Misère Tic-Tac-Toe on Projective Binary Steiner Triple Systems

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Abstract. Imagine playing tic-tac-toe to lose. Two players, Xavier and Olivia, alternate marking squares as usual. As soon as one player owns three squares in a row, they *lose*. The combinatorial game "misère tic-tac-toe" generalizes this idea. The two players must first agree on a board made from points and lines, which are subsets of the points—but this need not be a traditional tic-tac-toe board. In this article, we study misère tic-tac-toe on projective binary Steiner triple systems. We provide an explicit winning strategy for the second player, Olivia. This winning strategy relies on the nested geometric structure of these systems, as well as the structure of caps within them. This article completes the final case for misère tic-tac-toe on the "geometric" Steiner triple systems, with the surprising result that the winning strategy belongs to different players on affine versus projective Steiner triple systems.

1. INTRODUCTION. Find a friend and draw a tic-tac-toe board. Play as usual, except that the first player to get three in a row *loses*. It can be surprisingly hard to wrap your head around the new strategies you must use in this game! This "playing to lose" variation of tic-tac-toe is called **misère tic-tac-toe**.

You don't have to play this game on a standard tic-tac-toe board, however. Misère tic-tac-toe, also known as "reverse tic-tac-toe" or "toe-tac-tic," is a two-player combinatorial game. The players begin by choosing a finite board consisting of points and lines (each of which is a subset of the points). They alternate taking points until one "owns" all of the points on a line—and immediately loses. In traditional tic-tac-toe, the points are the boxes where players can write "X" or "O" (hence taking that point), and the lines are any of the three-in-a-rows that cause a player to lose. We represent this board using points and lines in Figure 1.

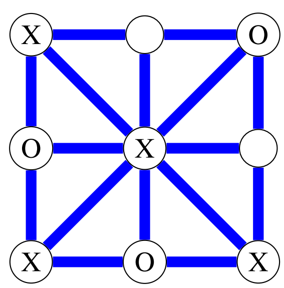


Figure 1. A misère tic-tac-toe board represented as points and lines, with a loss for X.

A board made from an arbitrary set of points and lines (which are subsets of the points) is called a **hypergraph**. Two-player games played on hypergraphs that have no hidden information and no random elements, such as misère tic-tac-toe, are called **positional games**. For a detailed survey of positional games, see [**9**]. "Normal" tic-tac-toe on hypergraphs (in which the first player to complete a line is the winner) has been especially well studied, and the winning player is known in many cases [**1**].

However, these are often existence results: There are few explicit strategies known for the winning player.

In normal tic-tac-toe on hypergraphs, either the first player has a winning strategy, or both players can force a draw [**9**]. There is no such result when we reverse the goal: In misère tic-tac-toe, either player could have a winning strategy depending on the board chosen, or both may be able to force a draw.

Thus, it is our goal is to find explicit strategies that tell one player exactly how to win misère tic-tac-toe. This has been done for a number of special classes of boards that have some additional structure.

One type of board that you might have already thought of is a cube, with lines running through rows, columns, and all diagonals. This generalizes to the "3^d-cell," in which the first player has a known winning strategy for certain values of d, and can tie for others [**8**].

Another misère tic-tac-toe board is inspired by the card game SET. This game is played with a special 81-card deck, in which each card has four attributes, with three values possible for each attribute. A "set" in the game consists of three cards for which each attribute either has the same value on each card, or else all three cards have different values. To play misère tic-tac-toe on SET, players alternate choosing any card from the deck. They lose if they ever hold three cards that form a "set" in their hand. Hence, the cards are points, and the "sets" are lines. The underlying geometric structure that these form is called a ternary affine geometry [**7**]. In the smallest case, playing misère tic-tac-toe with nine cards that have two of their attributes held constant is equivalent to playing "torus tic-tac-toe" to lose. In [**5**], the authors give a simple winning strategy for the first player on these boards.

The ternary affine geometries of SET are examples of a combinatorial object called a Steiner triple system or STS. Steiner triple systems share many of the key geometric features with traditional tic-tac-toe boards: They have three points per line, and each pair of points appears on exactly one line. Thus STSs are natural boards for misère tic-tac-toe.

In this article, we study misère tic-tac-toe on another type of STS, the projective binary Steiner triple systems with $2^n - 1$ points, and give an explicit winning strategy for the second player. This strategy relies on the nested geometric structure of these systems. Together with affine ternary STSs, these STSs are known as the "geometric" STSs. Our results answer the question of winning strategies for misère tic-tac-toe on the geometric STSs. To the best of our knowledge, explicit winning strategies for misère tic-tac-toe on Steiner triple systems are unknown beyond these geometric STSs.

We begin by providing background information on Steiner triple systems in Section 2. In Section 3, we consider the smallest nontrivial Steiner triple system which forms a basis for our subsequent work. We describe the winning strategy in Section 4. Finally, in Section 5, we summarize our results and propose open problems related to our work. Proofs of some technical results are contained in the Appendix (Section 6).

2. BACKGROUND. In this section, we define the key objects that we will study in this article: Steiner triple systems, which generalize the key features of tic-tac-toe boards. A **Steiner triple system on n points**, or **STS(n)**, is a set of n objects $\mathcal{P}$ called **points** together with a set $\mathcal{L}$ of subsets of $\mathcal{P}$ called **lines**, such that

1. Every line consists of 3 points, and
2. Every pair of points is contained in exactly one line.

There are a huge variety of STSs (see [**2**, Section 1] for additional information). Indeed, there are 80 nonisomorphic STS(15)s. Thus, we focus on a particular type of

STS. When $n = 2^k - 1$ $(k \geq 3)$, we can construct a **projective binary STS(n)** or PBSTS(n). The points are binary vectors of length k, excluding the zero vector. Lines are subsets of 3 points whose binary sum is $\vec{0}$. That is,

$$\mathcal{P} = \mathbb{F}_2^k \setminus \{\vec{0}\} \text{ and}$$

$$\mathcal{L} = \{\{a, b, c\} : a, b, c \in \mathcal{P} \text{ and } a + b + c \equiv \vec{0} \pmod 2\}.$$

Below, we verify that this construction does indeed produce an STS.

Theorem 1. *Every PBSTS(n) is a Steiner triple system.*

Proof. We show that the two conditions in the definition of STS are met:

1. By definition, a line $\{a, b, c\}$ of a PBSTS(n) could only fail to contain three points if two points are equal, say $a = b$. In this case $a + b + c \equiv 2a + c \equiv \vec{0} \pmod 2$, which implies that $c \equiv \vec{0} \pmod 2$. As $\vec{0}$ is not a point, this is impossible, and so each line consists of exactly three points.
2. Given two points a and b, the point $a + b$ lies on a line with them. This can be seen because $a + b + (a + b) \equiv 2a + 2b \equiv \vec{0} \pmod 2$. If $a + b + c \equiv \vec{0} \pmod 2$, then $c \equiv -a - b \equiv a + b \pmod 2$. Thus a and b are contained in exactly one line.

Thus every PBSTS(n) satisfies the definition of Steiner triple system. ■

This article will only address PBSTSs, and so all addition will be done modulo 2. Thus we will not write "mod 2" in subsequent calculations.

PBSTSs have been well studied. An alternative construction for them uses the points and lines of binary projective geometries, which are a type of finite geometry. Most of the terminology introduced in this section comes from this finite geometry context and applies equally well to other finite geometries. See [**2, 6**] for further details.

Example. Here we construct a PBSTS($2^3 - 1$). The points are binary vectors of length 3 except for the all-zeros vector, where we abbreviate the notation for the point (a, b, c) as abc:

$$\mathcal{P} = \{001, 010, 011, 100, 101, 110, 111\}.$$

Lines are subsets of three points whose vector sum is 000. For example, $001 + 101 + 100 = 000$, and so $\{001, 101, 100\}$ is a line. The complete list of lines is:

$$\{001, 010, 011\}, \quad \{010, 100, 110\}, \quad \{011, 101, 110\},$$
$$\{001, 100, 101\}, \quad \{010, 101, 111\}, \quad \{011, 100, 111\},$$
$$\{001, 110, 111\}.$$

We illustrate this PBSTS(7) by connecting the labeled points with lines, as in Figure 2. Note that line $\{010, 100, 110\}$ is drawn as a circle here. The only relevant property of a line is that the three points on it sum to 000—it need not be "straight"!

In misère tic-tac-toe, the game ends immediately once one player owns a line. Hence, any winning strategy must involve sets of points that do not completely contain a line. Luckily, this is a very well studied concept: A **cap** is a set of points in an STS

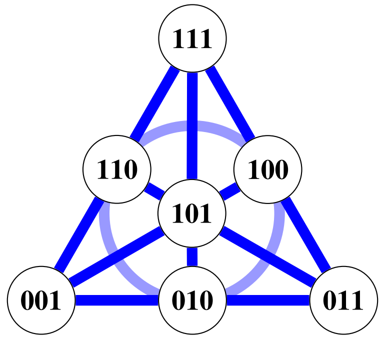


Figure 2. PBSTS(7).

that does not contain any lines. A **maximum cap** is a cap having the largest possible size for the given STS.

The size of a maximum cap is a hard limit on how long the game can continue: Once one player owns all points in a maximum cap, that player *must* lose on their next turn (unless, of course, the other player loses first). Conveniently, we know exactly how large a cap can be in PBSTSs.

Proposition 2 (see [3]). *The maximum size of a cap in a PBSTS(*$2^k - 1$*) is* 2^{k-1}*, and such a cap exists.*

Example. In the unique PBSTS(7), the points $\{100, 101, 110, 111\}$ form a cap. Notice that no line in Figure 2 is entirely contained in these points. By Proposition 2 these 2^{3-1} points must therefore be a maximum cap.

Maximum caps in PBSTSs are not rare, and in fact there is an easy construction. Take the set of all points in a PBSTS$(2^k - 1)$ whose leftmost coordinate is a 1. The sum of any three of these points will never be $\vec{0}$, because their sum will have $1 + 1 + 1 = 1$ in the leftmost coordinate. Thus no three such points can form a line. There are 2^{k-1} of these points, and so this cap is maximum. This is just one maximum cap—in general, there are many distinct maximum caps within each PBSTS.

PBSTSs have a nested structure. A **subsystem** of a PBSTS($2^k - 1$) is a subset of the points and lines that form a PBSTS($2^j - 1$) with $j \leq k$. If the subsystem has size $2^{k-1} - 1$ (the largest size smaller than the original PBSTS) it is called a **hyperplane**.

For convenience, we allow a line to be a hyperplane of a PBSTS(7). Subsystems are essentially vector subspaces without the zero vector. Hence, if we have two points in a subsystem, the unique third point that forms a line with them is also within the subsystem, and so the whole line is within the subsystem.

We will often speak of the **subsystem defined by a set of points**. This is the smallest subsystem that contains all of the points. For example, within a PBSTS(7), any two points define a line, while any three noncollinear points define the full PBSTS(7).

There is a key relationship between hyperplanes and maximum caps that we will use frequently in this article:

Proposition 3 (Segre [11]). *For any maximum cap* M *of a PBSTS with point set* P*,* $P \setminus M$ *is a hyperplane of the PBSTS.*

Example. Consider the PBSTS(15) S whose points are the 15 vectors $\mathcal{P} = \mathbb{F}_2^4 \setminus \{0000\}$. Then S has 35 lines. Consider the subset $\mathcal{P}'$ of those points whose leftmost coordinate is 0, that is, points of the form $0abc$. We have

$$\mathcal{P}' = \{0001, 0010, 0011, 0100, 0101, 0110, 0111\}.$$

Because the leftmost coordinate of these points is 0, three points in $\mathcal{P}'$ form a line if and only if their three rightmost coordinates sum to 0. This gives these seven lines:

$$\{0001, 0010, 0011\}, \quad \{0010, 0100, 0110\}, \quad \{0011, 0101, 0110\},$$
$$\{0001, 0100, 0101\}, \quad \{0010, 0101, 0111\}, \quad \{0011, 0100, 0111\},$$
$$\{0001, 0110, 0111\}.$$

These are exactly the same lines as the PBSTS(7), with an extra 0 added on the left of each point. Thus the points in $\mathcal{P}'$ and these 7 lines form a subsystem S' of size 7 within the PBSTS(15). This subsystem S' is a hyperplane of S, since S' is a PBSTS($2^3 - 1$) within a PBSTS($2^4 - 1$).

Further, to illustrate Proposition 3, consider the set of points

$$M = \mathcal{P} \setminus \mathcal{P}' = \{1001, 1010, 1011, 1100, 1101, 1110, 1111, 1000\}.$$

It can be verified that the 8 points in M are a cap of S, and by Proposition 2 they are in fact a maximum cap. As we have seen, the points $\mathcal{P} \setminus M = \mathcal{P}'$ do indeed form a PBSTS(7), which is a hyperplane of S.

The PBSTS(15) and substructures are illustrated in Figure 3. The points of $\mathcal{P}'$ appear on the left and are illustrated as a hyperplane isomorphic to PBSTS(7). The middle set of points are the same as those in $\mathcal{P}'$ with the leftmost coordinate set to 1, and are shown in corresponding positions. We label these middle points with the notation $\mathcal{P}' + 1000$. One new point, 1000, has no analog in $\mathcal{P}'$ and appears on the far right. The maximum cap M consists of all eight points on the right side. Note that many lines have been left out of this figure; however, all such lines must have at least one point in $\mathcal{P}'$. Three examples of such lines are shown (dashed).

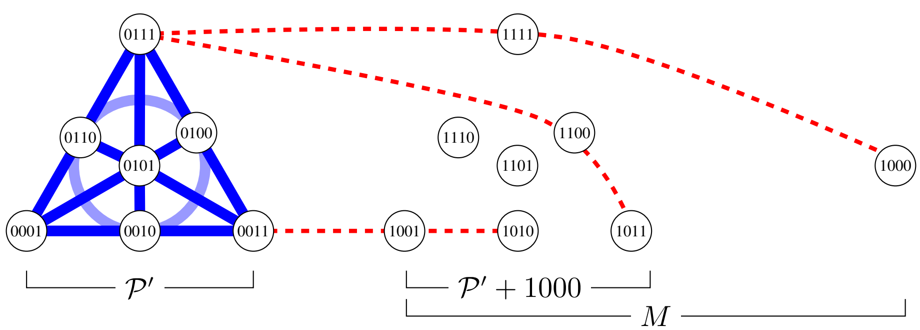


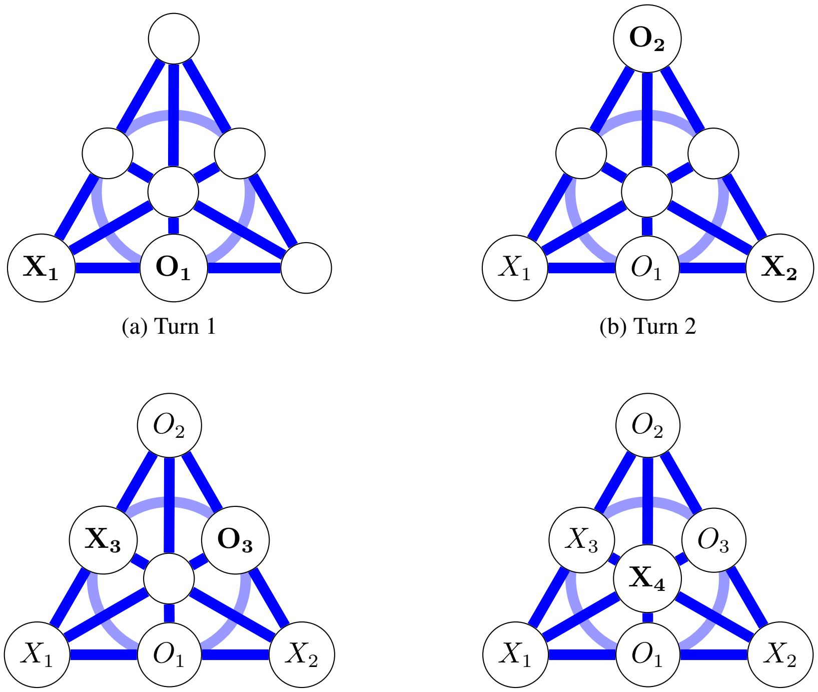
Figure 3. PBSTS(15).

3. MISÈRE TIC-TAC-TOE AND THE FANO PLANE.

The PBSTS(7) in Figure 2, known as the **Fano plane**, is the smallest nontrivial STS and is the unique STS of its size. The Fano plane is the simplest place to play misère tic-tac-toe on an STS. In this section, we introduce the rules and notation used in misère tic-tac-toe. We study the game on the PBSTS(7), which will form the basis for our subsequent general strategy.

To play misère tic-tac-toe on an STS, two players take turns selecting points until one of them has chosen all three points in a line, losing immediately. We name the first player Xavier, and the second player Olivia. A **move** is a single play by either player. We denote Xavier's and Olivia's ith moves by X_i and O_i, respectively, with $i \geq 1$. A **turn** is a pair of moves beginning with Xavier: (X_i, O_i).

Example. We give an example of gameplay on the Fano plane, illustrated in Figure 4.



(a) Turn 1 (b) Turn 2 (c) Turn 3 (d) Turn 4, Xavier's move

Figure 4. Example of gameplay.

Turn 1: Xavier begins choosing X_1 arbitrarily. Olivia follows up with O_1, again arbitrarily chosen, since every point forms a unique line with X_1.

Turn 2: Xavier chooses his second move, X_2, to complete the unique line formed by X_1 and O_1. This guarantees that this line can never cause him to lose. Olivia realizes that every choice of points is essentially the same for her: Each unclaimed point will form a distinct line with each of the three claimed points, and each of those lines has only one point already taken. So, O_2 is again arbitrary.

Turn 3: Xavier chooses X_3 so that he completes a line containing one of Olivia's points—a safe move. Olivia now has two choices, one of which would cause her to lose. She lets O_3 be the other point.

Turn 4: Xavier has no choice but to select the center point to be X_4, which causes him to lose with line $\{X_2, X_3, X_4\}$.

The previous example illustrates a key result about misère tic-tac-toe on the Fano plane that will form the basis for gameplay on larger PBSTSs.

Theorem 4 (PBSTS(7) Strategy). *The second player, Olivia, wins misère tic-tac-toe on the PBSTS(7) by following this strategy:*

1. *Olivia's first two moves are arbitrary.*
2. *Olivia's third move is any point that is not $O_1 + O_2$.*

Proof. Since Olivia plays second, she takes at most three points, and she can only lose by forming a line with those three points. Olivia's first two moves O_1 and O_2 define a unique line $\{O_1, O_2, O_1 + O_2\}$, and so only the point $O_1 + O_2$ can cause her to lose. Just before Olivia's third move, there are two unclaimed points remaining, at most one of which is $O_1 + O_2$. Thus Olivia can always select a nonlosing point. Xavier then takes the seventh and final point in the game. It is known that for every partition of the points of the PBSTS(7) into sets of three and four points, one set must contain a line [**4**], and hence the game cannot end in a tie. Since Olivia can't lose or tie, she wins. ■

Notice that in the above strategy, Xavier takes four points—the size of a maximum cap in the PBSTS(7)—but these points are *not* a cap since they contain a line.

4. A WINNING STRATEGY ON ALL PROJECTIVE BINARY STEINER TRIPLE SYSTEMS. In this section, we develop a winning strategy for all PBSTSs. Not to give too much away, but Olivia—the second player—wins. This is a bit surprising, since in [**5**] the authors found a winning strategy for Xavier—the first player—on a different set of STSs. The STSs studied in [**5**] are known as affine ternary STSs, and are structurally related to the card game SET. Together, PBSTSs and affine ternary STSs are known as the "geometric" STSs, due to their connections with finite geometry.

We start by establishing a simple condition that, if Olivia can meet it, guarantees her win. Throughout, we assume that $k \geq 3$.

Lemma 5. *In misère tic-tac-toe played on the PBSTS($2^k - 1$), if Olivia can obtain $\frac{3}{4}$ of the points in a maximum cap, then Xavier must lose on his next move.*

Proof. By Proposition 2, a maximum cap M has 2^{k-1} points, so $\frac{3}{4}$ of the points is

$$\frac{3}{4}(2^{k-1}) = 3 \cdot 2^{k-3} = 2 \cdot 2^{k-3} + 2^{k-3} = 2^{k-2} + 2^{k-3}.$$

Because these points are in a cap, Olivia cannot form a line by taking them.

We now count the maximum number of points that Xavier may take. Within M, Xavier could take at most the 2^{k-3} points not taken by Olivia. All of Xavier's other moves must come from outside M. By Proposition 3, the $(2^k - 1) - 2^{k-1} = 2^{k-1} - 1$ points outside of M form a hyperplane H that is a PBSTS($2^{k-1} - 1$).

By Proposition 2 a maximum cap in H contains 2^{k-2} points. If Xavier takes more than this number of points within H, he will lose. Thus Xavier can take at most 2^{k-3} points from M and at most 2^{k-2} points not in M (in H), for a total of $2^{k-2} + 2^{k-3}$ points. This is the same number of points that Olivia is assumed to take.

Thus at the end of turn $2^{k-2} + 2^{k-3}$, both players have taken $2^{k-2} + 2^{k-3}$ points. Because Xavier is the first player, he will play next. By the previous argument, he must take a point that will complete a line and so will lose. ■

In the remainder of the article we will show how Olivia can obtain the condition in Lemma 5. Many of our key results rely on Olivia identifying a maximum cap within a PBSTS. To identify maximum caps, we must better understand the nested structure of both subsystems and caps.

Given a subsystem S of a PBSTS($2^k - 1$), a **peak** relative to S is the first point taken by Xavier that falls outside S. If $q \in S$, then the **lift** of q is $p + q$. Note that $p + q \notin S$. A point, its lift, and the peak form a line.

Suppose M is a subset of the points of a subsystem S with peak p. We use the convenient shorthand $M + p$ to denote the set of all lifts of the points in M. That is, $M + p = \{x + p : x \in M\}$. Using the vector structure of these PBSTSs, if a subsystem S is a PBSTS($2^j - 1$) with point set M and p is a peak, then $M \cup (M + p) \cup \{p\}$ together with all appropriate lines form a PBSTS($2^{j+1} - 1$). This shows how subsystems can "grow" in a recursive way. Note that each subsystem has a unique peak, and as Xavier chooses peaks, he creates larger and larger subsystems, until eventually no further peaks are available.

A peak, lifts, and the related notation are illustrated in Figure 3: The subsystem S with points $\mathcal{P}'$, which is isomorphic to the PBSTS(7), has peak 1000 on the far right. The lift of each point in $\mathcal{P}'$ appears to its right in the area labeled $\mathcal{P}' + 1000$. The entire PBSTS(15) has point set $\mathcal{P}' \cup (\mathcal{P}' + 1000) \cup \{1000\}$.

Every PBSTS(n) has a **replication number** $r = \frac{n-1}{2}$ that represents the number of lines containing each point. This value is independent of the structure of the STS.

Lemma 6. *Every line that intersects a maximum cap M of a PBSTS($2^k - 1$) intersects M in exactly two points.*

Proof. Consider a point $p \in M$. There are $r = 2^{k-1} - 1$ lines through p, and $2^{k-1} - 1$ points other than p in M. Each point $p \neq x \in M$ defines a unique line with p. None of these lines contains a third point in M, since M contains no complete lines. This accounts for all $2^{k-1} - 1$ lines through p. Thus every line through p intersects M in exactly two points. ■

A maximum cap can be "grown" from a smaller cap in a recursive way that we describe next. This will allow Olivia to determine a well-defined maximum cap of a PBSTS as the game progresses.

Lemma 7. *Let M be a maximum cap in a subsystem S of size $2^j - 1$ within a PBSTS($2^k - 1$), where $j < k$. Suppose p is the peak relative to S. Then $M' = M \cup (M + p)$ is a maximum cap of the subsystem S' defined by S and p.*

Proof. By Proposition 2, M contains 2^{j-1} points. By definition $M \subseteq S$, while every point of $M + p$ is in $S' \setminus S$, and thus M and $M + p$ are disjoint. Therefore $|M'| = 2^{j-1} + 2^{j-1} = 2^j$, the correct size for a maximum cap of the PBSTS($2^{j+1} - 1$) defined by S and p.

It remains to show that M' does not contain any lines. Suppose to the contrary that three points $\{x, y, z\}$ of M' do form a line. We consider cases based on the number of these points that are in M.

Case 3: We cannot have $\{x, y, z\} \subseteq M$ because M is a cap.

Case 2: Suppose $x, y \in M$. Then we can write $z = a + p$ for some point $a \in M$. Thus $0 = x + y + z = x + y + (a + p) = (x + y + a) + p$. Thus $x + y + a = p$. However, $x + y + a \in S$ by definition of a subsystem, and $p \notin S$, a contradiction.

Case 1: Suppose $x \in M$. Then we can write $y = b + p$ and $z = a + p$ for some points $a, b \in M$. Thus $0 = x + y + z = x + (b + p) + (a + p) = x + b + a$. This implies that $\{x, b, a\}$ is a line in M, contradicting the fact that M is a cap.

Case 0: Here we can write $x = c + p, y = b + p$, and $z = a + p$ for some $a, b, c \in M$. Thus $0 = x + y + z = (c + p) + (b + p) + (a + p) = (a + b + c) + p$. This leads to the same contradiction as Case 2.

Therefore, M' is indeed a maximum cap of S'. ■

Maximum caps can be partitioned into useful subsets called tiles. Given a maximum cap M, a point $a \in M$, and points $p, q \notin M$, the **tile** containing a is $S_{p,q}(a) = \{a, a+p, a+q, a+p+q\}$. We can visualize a tile as the box in Figure 5.

Lemma 8. *Given a maximum cap M of a PBSTS($2^k - 1$) and two points $p, q \notin M$, M can be partitioned into 2^{k-3} disjoint tiles $\{S_{p,q}(a) : a \in M\}$.*

Proof. We will show that each point a of M is in exactly one tile $S_{p,q}(a)$, and that each $S_{p,q}(a) \subseteq M$.

Each point $a \in M$ is in the tile $S_{p,q}(a) = \{a, a+p, a+q, a+p+q\}$. Because $p, q \notin M$, by Lemma 6 we have $a+p$, $a+q \in M$. Since $a+p \in M$ and $q \notin M$, by the same lemma we have that $a+p+q \in M$. Therefore, each $S_{p,q}(a) \subseteq M$.

Now consider $a \in S_{p,q}(a)$ and suppose also $a \in S_{p,q}(b)$ for some b. Then we can write $a = b + x$ where x is either $\vec{0}$, p, q, or $p+q$. Thus $b = a + x$, and so $b \in S_{p,q}(a)$. A similar statement holds for all other elements of $S_{p,q}(a)$ and $S_{p,q}(b)$, and therefore $S_{p,q}(a) = S_{p,q}(b)$. This shows that each point of M can only be in one tile.

Therefore, we have a partition of M into disjoint tiles of 4 points each. By Proposition 2, $|M| = 2^{k-1}$, and so the number of tiles is $\frac{2^{k-1}}{4} = 2^{k-3}$. ∎

We are now ready to show how tiles and caps together limit Xavier's possible plays.

Lemma 9. *Suppose Xavier and Olivia play misère tic-tac-toe on the PBSTS($2^k - 1$). Let M be a maximum cap of this PBSTS and suppose Xavier holds points $p, q \notin M$. If Xavier chooses some point $a \in M$, then the only other point that Xavier may choose in $S_{p,q}(a)$, without immediately losing, is $a+p+q$.*

Proof. The four points in $S_{p,q}(a)$ are $a, a+p, a+q, a+p+q$. Because Xavier holds a as well as p, we have that $a+p$ would form line $\{a, p, a+p\}$. A similar statement holds for $a+q$. This leaves only $a+p+q$. ∎

Lemma 9 is illustrated in Figure 5. This configuration of points and lines is known as the **Pasch configuration**, and is well studied. In particular, PBSTSs contain the maximum possible number of Pasch configurations among all STSs [**6**].

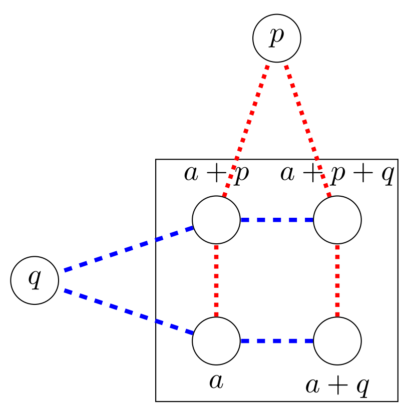


Figure 5. The Pasch configuration with tile $S_{p,q}(a)$.

Lemma 9 shows that Xavier can take at most two points in every tile of a maximum cap M. This suggests an important strategy for Olivia: Whenever Xavier takes one point $a \in M$, Olivia immediately takes $a + p + q$, which is Xavier's only other possible point within that tile. By doing this, Olivia limits Xavier to exactly one point out of every four in the cap. If Olivia takes only points within this maximum cap, then Olivia can obtain at least three out of every four points—that is, $\frac{3}{4}$ of the points in M. If she can do this, then by Lemma 5, Olivia will win.

Of course, this is a wonderful idea—if it can be done! There are many questions that we will have to answer first, including: Can Olivia always take the desired point? And, more fundamentally: How is this cap M even defined?

We begin to answer these questions, and hence prove that Olivia can win, with the following lemmas. The proofs of Lemmas 10 through 12 are fairly technical and do not add much to the exposition, so we have placed their proofs in the Appendix and instead illustrate each result with figures.

Lemma 10. *Suppose Xavier and Olivia play misère tic-tac-toe on a PBSTS($2^k - 1$). Within the first two turns of the game, Olivia can ensure that all of these occur:*

1. *The players' moves define a unique PBSTS(7), P.*
2. $O_1 = X_1 + X_2$.
3. $O_2 \in P$.

Lemma 10 assures us that the game always starts in the same basic way, regardless of Xavier's choices. After two turns, without loss of generality, the players' moves define a PBSTS(7) illustrated in Figure 6 referred to as the **base plane** of the game.

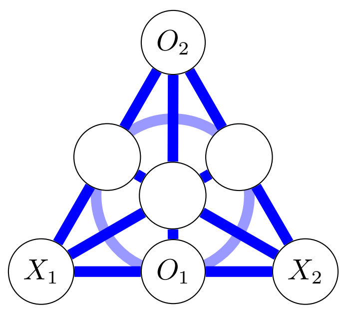


Figure 6. The first two turns of the game.

Now, we study the next two turns of the game. Beyond these first four turns, Olivia will be able to implement her more general strategy outlined above.

Lemma 11. *Suppose Xavier and Olivia play misère tic-tac-toe on a PBSTS($2^k - 1$). Within the first four turns of the game, Olivia can ensure that all of the following occur:*

1. *Xavier has taken a peak p_1.*
2. *Olivia owns the points $O_1 + p_1$ and $X_1 + O_2$.*
3. *There exists a well-defined maximum cap M' of eight points within the system defined by the base plane and p_1. Furthermore, $X_1, p_1 \notin M'$.*
4. *Xavier can own at most one point in the tile $S_{X_1,p_1}(X_2)$. If $X_4 \in M'$, then Xavier can own at most one point in the tile $S_{X_1,p_1}(X_4)$.*

After four turns, Lemma 11 guarantees us a certain structure, illustrated in Figure 7. The Fano plane on the left is the base plane. The points on the right are the lifts of the corresponding points on the left. Note that there are many lines *not* drawn that connect base and lift points. Xavier's moves are labeled with X's, as well as p_1. Olivia's moves are labeled with O's as well as $O_1 + p_1$ and $X_1 + O_2$. Points in the cap M' have thick outlines. Note how each point in the base plane's cap has a corresponding lifted point that is also in M'. Dashed thick strokes denote $S_{X_1,p_1}(X_2)$. Note that Xavier cannot take $X_2 + p_1$, the lift of X_2, since it would form a line for him. Thus this point is marked (O) as it is safe for Olivia to take in the future. One of Xavier's moves is not shown—it could be one of the points not taken, or even another peak not in this diagram.

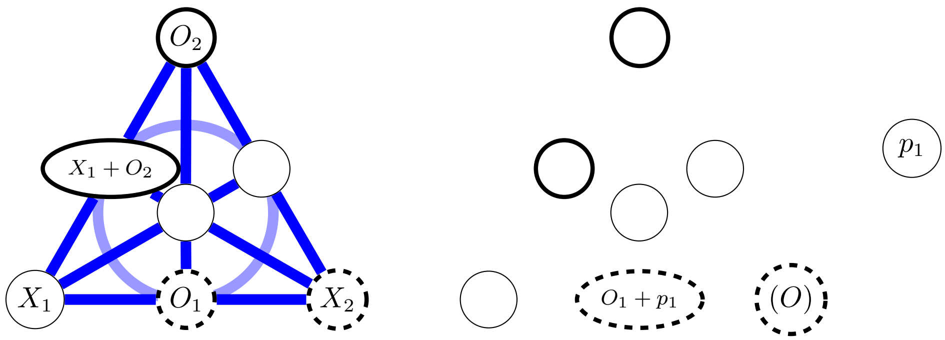


Figure 7. The first four turns of the game.

One consequence of Lemma 11 is that Olivia can identify a maximum cap as of the fourth turn of the game. Next, Lemma 12 shows that as the players continue to play, Olivia can always identify a maximum cap within the subsystems that their moves define. Furthermore, the partition into tiles that Lemma 8 guarantees is "stable" in the sense that as each cap grows, the existing tiles remain unchanged.

Lemma 12. *Suppose Xavier and Olivia play misère tic-tac-toe on the PBSTS(*$2^k - 1$*). Let* X_1 *be Xavier's first move, and let* p_1 *be the first peak that Xavier takes. At every turn* i *in the game with* $i \geq 4$*, Olivia can determine a maximum cap* M_i *of the subsystem defined by all previous moves such that for all* i*,* $M_{i-1} \subseteq M_i$*,* $X_1 \notin M_i$*, and* $p_1 \notin M_i$*. Furthermore, let* T_{i-1} *and* T_i *be the partitions into tiles of* M_{i-1} *and* M_i*, respectively, guaranteed by Lemma 8. Then* $T_{i-1} \subseteq T_i$*.*

We inch closer to a winning strategy by showing that Olivia's "$\frac{3}{4}$ win condition" from Lemma 5 can also force Xavier to take peaks at certain points during the game.

Lemma 13. *Suppose Xavier and Olivia play misère tic-tac-toe on the PBSTS(*$2^k - 1$*), and let maximum caps* M_i *be defined as in Lemma 12. If, on turn* $i \geq 4$ *Olivia obtains* $\frac{3}{4}$ *of the points in* M_i*, then on turn* $i + 1$ *either Xavier takes a peak, or Xavier loses.*

Proof. Consider the subsystem defined by all moves up through X_i. This is a PBSTS($2^j - 1$) for some $j \leq k$. If this subsystem is not maximal—that is, if $j < k$—then Xavier may take a move from outside the current subsystem, which will then be a peak. If Xavier takes a move from within the subsystem, then by Lemma 5 he will lose immediately. ■

Lemma 13 shows that Xavier must periodically "escape" from the subsystem containing current points. This guarantees that the caps M_i keep expanding as described in Lemma 7, giving Olivia more room to play safely.

At long last, we are ready to show how Olivia can win the game, regardless of Xavier's moves.

Theorem 14. *Olivia has a winning strategy for misère tic-tac-toe played on any projective binary STS.*

Proof. Suppose Xavier and Olivia play misère tic-tac-toe on the PBSTS($2^k - 1$). We will show that, regardless of Xavier's choices, either Xavier makes a losing move, or Olivia can obtain $\frac{3}{4}$ of the points in a maximum cap M of the PBSTS($2^k - 1$), and hence she will win by Lemma 5.

Lemmas 10 and 11 describe Olivia's moves for the first 4 turns of the game. By Lemma 12, for all turns $i \geq 4$ Olivia has a well-defined maximum cap M_i of the subsystem spanned by the players' moves. Furthermore, X_1 and p_1 are not in M_i, where p_1 is the first peak taken by Xavier.

By Lemma 8, M_i can be partitioned into disjoint tiles relative to p_1 and X_1. In addition to being outside M_i, X_1 and p_1 are both owned by Xavier.

Suppose that on turn $i > 4$ Xavier chooses a point $X_i = a \in M_i$. Xavier has therefore chosen a point in $S_{X_1,p_1}(a)$. If it is available, Olivia selects $O_i = a + X_1 + p_1$. If this point is not available, or if Xavier chooses $X_i \notin M_i$, then Olivia selects any point in M_i that has not been taken.

We note that if $a + X_1 + p_1$ is not available for Olivia to choose, then Olivia must have chosen it earlier in the game. If it had been selected by Xavier, by this strategy, Olivia would have claimed $(a + X_1 + p_1) + X_1 + p_1 = a$ immediately after Xavier chose $a + X_1 + p_1$. Then a would not have been available for Xavier to choose on the current turn. Thus, at the end of turn i, Olivia owns $a + X_1 + p_1$ and $O_i \in M_i$.

Further, Olivia is always able to take a point on her turn. Because Olivia always owns $a + X_1 + p_1$ within the tile where Xavier just played, by Lemma 9 Xavier can own at most one point in each tile of M_i. Thus if there were no point available for Olivia to take in M_i, it must be the case that as of turn $i - 1$, Olivia owned at least three points in each tile of M_{i-1}, and so at the end of turn $i - 1$ Olivia satisfied the hypotheses of Lemma 13. Thus Xavier's move X_i either caused him to lose the game immediately (in which case Olivia celebrates her win rather than making a move), or else X_i was a peak. In the latter case, by Lemma 12 $M_i = M_{i-1} \cup (M_{i-1} + X_i)$, and none of the points in $M_{i-1} + X_i$ have been taken. Thus Olivia can take any point in $M_{i-1} + X_i$.

In all cases, all of Olivia's moves are in the cap M_i, and hence Olivia can't lose. By this strategy, on some turn j Olivia defines a maximum cap $M = M_j$ of the PBSTS($2^k - 1$). Olivia will own at least three points in every tile of M (including, by Lemma 11, in each of $S_{X_1,p_1}(X_2)$ and, if $X_4 \in M$, $S_{X_1,p_1}(X_4)$ as well).

Therefore, if Xavier does not make an earlier losing move, Olivia will eventually obtain at least $\frac{3}{4}$ of the points in M, and by Lemma 5, Xavier will lose on his next turn. Thus Olivia wins. ■

The final result of this strategy is shown schematically in Figure 8. Within the maximum cap M, Olivia is able to obtain three out of four points in every tile, thus guaranteeing her the win condition from Lemma 5. The details of the starting moves that allow Olivia to do this, which are addressed in Lemmas 10 through 12, are not shown.

5. SUMMARY AND OPEN PROBLEMS.

This article presents an explicit winning strategy for the second player, Olivia, for misère tic-tac-toe played on all projective binary Steiner triple systems. An explicit winning strategy for Xavier has been previ-

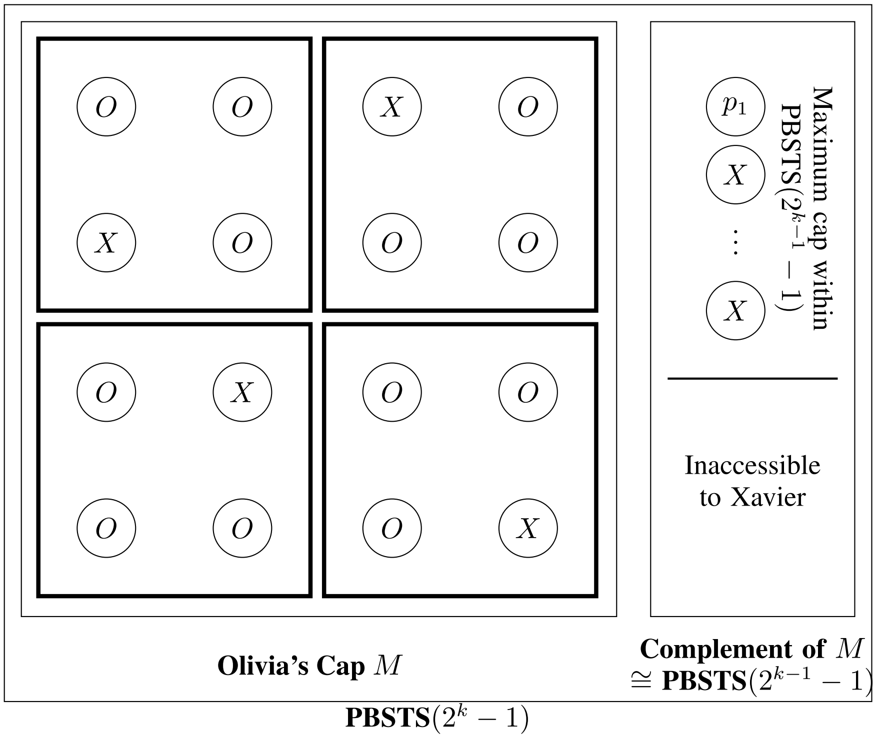


Figure 8. The end result of playing Olivia's winning strategy. Tiles are outlined in thick squares.

ously found on affine ternary Steiner triple systems [**5**]. Together, these articles determine winning strategies for misère tic-tac-toe on all geometric Steiner triple systems.

Steiner triple systems are a natural structure on which to play misère tic-tac-toe. The fact that each line has three points echoes tic-tac-toe, as well as the fact that every pair of lines intersects in at most one point. There are many Steiner triple systems beyond the ones studied so far, and so we suggest studying misère tic-tac-toe on Steiner triple systems that are not built from finite geometries.

In this article, the recursive structure of the PBSTSs played a key role. The famous "doubling construction" creates new STSs from smaller ones in a similar recursive process. We suspect that some key results from this article, such as the ability to identify and track a maximum cap through a sequence of nested subsystems, would work in doubled STSs as well.

An interesting variant on misère tic-tac-toe, "notakto," involves both players marking the board with X's [**10**]. The first player to complete a line loses. This game can easily be played on PBSTSs. However, the strategy presented in this article cannot be easily extended to notakto on a PBSTS, since Olivia's strategy depends heavily on completing lines where Xavier owns at least one point. We believe that finding an optimal notakto strategy on PBSTSs is a nontrivial task.

6. APPENDIX: PROOFS OF STRATEGY RESULTS. This appendix contains the proofs of several lemmas stated above. Because these proofs are fairly technical, we present them here for completeness and give only their statements in the main exposition.

Proof of Lemma 10. In any PBSTS, a line and a point not on the line define a subsystem which is a PBSTS(7). The first two moves X_1 and O_1 are arbitrary. Now, Xavier's second move X_2 can be one of two types of points:

Case 1: $X_2 = X_1 + O_1$. Thus all three points on one line of the PBSTS have been claimed. Olivia can select any available point to be O_2. The line and O_2 define a PBSTS(7) P. In addition, $X_2 = X_1 + O_1$ implies that $O_1 = X_1 + X_2$.

Case 2: $X_2 \neq X_1 + O_1$. Thus the first three moves do not form a line. This means that Olivia can choose the unclaimed point $O_2 = X_1 + X_2$. This completes a line in the PBSTS of which O_1 is not a point; hence O_1 and the line define a plane P. We may relabel O_1 and O_2 by swapping their labels, so $O_1 = X_1 + X_2$.

In either case, $O_1 = X_1 + X_2$ and $O_2 \in P$. ■

Proof of Lemma 11. We assume that the game is in the state guaranteed by Lemma 10, with base plane P. Thus we may begin considering the game at the start of turn three.

Our strategy will call for Olivia to take a point in P in one of turns three or four. Thus Xavier will be able to take at most one point in P. If Xavier were to take two points in P, then the plane would be full and Xavier would lose by Theorem 4. Thus, Xavier takes at least one point outside of P.

When Xavier first takes a point outside P, this is necessarily a peak p_1, proving point 1.

Olivia responds by taking $O_1 + p_1$ which must be available. Regardless of Xavier's other move, Olivia selects $X_1 + O_2$, unless this is the point that Xavier took. However, the system and all points are symmetric with respect to switching X_1 and X_2, so without loss of generality, we assume that Olivia takes $O' = X_1 + O_2$. This proves point 2.

Let $M = \{O_1, O_2, O', O_1 + O_2 + O'\}$. It is easy to check that this is a maximum cap of P. Notice that $O_1 + O_2 + O' = (X_1 + X_2) + O_2 + (X_1 + O_2) = X_2$. Thus, M does not contain X_1. Now define

$$M' = M \cup (M + p_1) = \{O_1, O_2, O', X_2, O_1 + p_1, O_2 + p_1, O' + p_1, X_2 + p_1\}.$$

By Lemma 7, M' is a maximum cap of the system defined by P and p_1 and hence contains eight points. By construction, $X_1 \notin M'$ and $p_1 \notin M'$. This proves point 3.

Finally, we address point 4. By Lemma 6, as $O_1 \in M'$ and $X_1 \notin M'$, we have that $X_2 = O_1 + X_1 \in M'$. Then

$$\begin{aligned} S_{X_1,p_1}(X_2) &= \{X_2, X_2 + p_1, X_2 + X_1, X_2 + p_1 + X_1\} \\ &= \{X_2, X_2 + p_1, O_1, O_1 + p_1\}. \end{aligned}$$

Olivia owns O_1 and $O_1 + p_1$ and Xavier cannot take $X_2 + p_1$ without forming a line. Next assume that $X_4 \in M'$ and consider the tile $S_{X_1,p_1}(X_4)$. Xavier owns X_4 and so cannot take $X_4 + X_1$ or $X_4 + p_1$. It remains to be shown that Xavier cannot own $X_4 + p_1 + X_1$. Consider M' as defined in point 3. All four of the points of M' in P have been taken before turn four, so if $X_4 \in M'$, then X_4 must have been a lift of a point in M. Thus $X_4 + p_1$ is that base point, so $X_4 + p_1 \in \{O_1, O_2, O', X_2\}$. This implies that $X_4 + p_1 + X_1 \in \{X_2, O', O_2, O_1\}$. Each of O_1, O_2, and O' were taken by Olivia before X_4, and so could not be taken by Xavier. If $X_4 + p_1 + X_1 = X_2$, then algebra gives us $X_4 = X_2 + X_1 + p_1 = O_1 + p_1$. This is a contradiction, since $O_1 + p_1$ is a point taken by Olivia (by point 2). Thus in no case can Xavier take $X_4 + p_1 + X_1$. ■

Proof of Lemma 12. By Lemma 11, by turn 4 Olivia can determine a maximum cap of eight points within a subsystem defined by the base plane and one peak. Recall that X_1 is not in this cap, and there are no peaks in it. This cap is M_4.

After turn four is completed, Xavier and Olivia continue taking points. For each move, Xavier either plays within the subsystem spanned by the current moves, or he takes a point from outside of this subsystem.

On turn i, if Xavier plays within the current subsystem, then Olivia sets $M_i = M_{i-1}$. That is, her cap remains the same.

If Xavier takes a point from outside the current subsystem, then his move is a peak p_i. In this case, Olivia sets $M_i = M_{i-1} \cup (M_{i-1} + p_i)$. By Lemma 7, M_i is a maximum cap of the subsystem defined by all previous moves together with p_i. Furthermore, $M_{i-1} + p_i$ is disjoint from the base plane, and so cannot contain X_1. By construction, p_1 is not added to this set either.

Finally, the partitions guaranteed by Lemma 8 depend only on the exterior points X_1 and p_1. Thus each tile $S_{X_1,p_1}(a) \subseteq M_{i-1}$ is also contained in M_i, so $T_{i-1} \subseteq T_i$. ■

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